
Correlated Equilibria for Approximate Variational Inference in MRFs

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Abstract

Almost all of the work in graphical models for game theory has mirrored previous work in probabilistic graphical models. Our work considers the opposite direction: Taking advantage of recent advances in equilibrium computation for probabilistic inference. In particular, we present formulations of inference problems in Markov random fields (MRFs) as computation of equilibria in a certain class of game-theoretic graphical models. While some previous work explores this direction, none of that work concretely establishes the precise connection between variational probabilistic inference in MRFs and correlated equilibria. There is no work that exploits recent theoretical and empirical results from the literature on algorithmic and computational game theory on the tractable, polynomial-time computation of exact or approximate correlated equilibria in graphical games with arbitrary, loopy graph structure. Our work discusses how to design new algorithms with equally tractable guarantees for the computation of approximate variational inference in MRFs. In addition, inspired by a previously stated game-theoretic view of state-of-the-art tree-reweighed (TRW) message-passing techniques for belief inference as zero-sum game, we propose a different, general-sum potential game to design approximate fictitious-play techniques. We perform synthetic experiments evaluating our proposed approximation algorithms with standard methods and TRW on several classes of classical Ising models (i.e., with binary random variables). Our experiments show that our global approach is competitive, particularly shinning in a class of Ising models with constant, “highly attractive” edge-weights, in which it is often better than all other alternatives we evaluated. While our more local approach was not as effective as our global approach or TRW, in fairness, almost all of the alternatives are often no better than a simple baseline: estimate the marginal probability to be 0.5.

1 Introduction

Almost all of the work in graphical games has borrowed heavily from analogies to probabilistic graphical models. However, over-reliance on those analogies and previous standard approaches to exact inference might have led that approach to face the same computational roadblocks that plagued most exact-inference techniques.

As is common knowledge within the graphical models community, exact inference is tractable in probabilistic graphical models whose graphs have bounded treewidth. Except for a few special cases such as Ising models with planar graphs without node biases, or with other very specific edge-weight properties,¹ the problem is for the most part considered intractable outside those confines [1, 3, 4].

¹ See Istrail [1] for a reasonably comprehensive account of the computational literature in this area up to the year 2000; several surveys of more recent advances, such as that of Wang et al. [2], are easily accesible and available online (<https://hal.inria.fr/hal-00858390v1/document>).

In 2005, Papadimitriou and Roughgarden [5] showed the intractability of computing the “social-welfare” optimum *correlated equilibria* (CE) in arbitrary graphical games (see also Papadimitriou and Roughgarden [6]). Everything seemed to point toward an eventual resignation that the approach of Kakade et al. [7], along with any other approach to the problem for that matter, had hit the “bounded-treewidth-threshold wall.”

Yet, soon after, Papadimitriou [8] took a radically different approach to the problem, and surprised the community with an efficient algorithm for computing CE not only in graphical games, but also in almost all known compactly representable games. Jiang and Leyton-Brown [9] built upon Papadimitriou’s idea to provide what most people would consider an improved polynomial-time algorithm, because of the simplification of the CE that their algorithm outputs (see also Jiang and Leyton-Brown [10] for a summary).²

An immediate question that arises from the algorithmic results just described is, what is so fundamentally different between the problem of exact inference in graphical models and equilibrium computation that made this result possible in the context of graphical games? Of course, CE, probabilistic inference, and their variants are different problems, even within the same framework of graphical models. The question is, how different are they?

It is well-known that *pure strategy Nash equilibrium* (PSNE) is inherently a classical/standard discrete *constraints satisfaction problem* (CSP). It is also well-known that any CSP can be cast as a most-likely, or equivalently, a *maximum a posteriori* (MAP) assignment estimation problem in *Markov random fields* (MRFs).³ Through this connection, it is clear that there exists a MAP formulation of PSNE. But what about other, more general forms of equilibria?

We present here a formulation of the problem of equilibrium computation as a kind of local conditions for different approximations to belief inference. Similarly, we show how one can view some special games, called *graphical potential games* [11], as defining an *equivalent* MRF whose “locally optimal” solutions correspond to *arbitrary* equilibria of the game. Hence, Papadimitriou’s result, and later that of Jiang and Leyton-Brown, open up the possibility that at least new classes of problems in probabilistic graphical models could be solved exactly and efficiently. The question is, which classes?

While we provide specific connections between the two fields that yield immediate theoretical and computational implications, we also provide practical alternatives that result from those connections. That is, the foundation of both Papadimitriou’s and Jiang and Leyton-Brown’s algorithms is the *ellipsoid method*, which is one approach that leads to the polynomial-time algorithm for linear programming. This approach, while provably efficient in theory, is often seen as less practical as other alternatives such as so-called *interior-point methods*. This is in contrast to the simple linear programs that are possible for certain classes of graphical games [7]. Are there simpler and practically effective variants of Papadimitriou’s or Jiang and Leyton-Brown’s algorithms? While the last question is an important open question, we do not address it directly in this paper. Instead, we employ ideas from the literature of learning in games [12], particularly no-regret algorithms and fictitious play, to propose two specific instances of game-theoretic inspired, practical, and effective heuristics for belief inference in MRFs, based on two different approaches: one local, the other global. We evaluate our proposed algorithms within the context of the most popular, standard, and state-of-art techniques from the literature in probabilistic graphical models.

This manuscript describes our work, which starts to address some of the questions above, and reports on our progress.

1.1 Overview of the Paper

Section 2 provides preliminary material, introducing basic notation, terminology, and concepts from graphical models and game theory.

²Papadimitriou’s work has an interesting history, which Jiang and Leyton-Brown [9] nicely summarize. Some questions aroused at the time about the technical soundness in the description of some steps in Papadimitriou’s algorithm. Jiang and Leyton-Brown [9] provided clarifications to those steps.

³Assuming a solution exists, of course; otherwise the resulting MRF is not well-defined.

Section 3 is the main technical section of the paper. It shows reductions of different problems in belief inference in MRFs as computing equilibria in graphical potential games compactly represented as *Gibbs potential games* [11]. The reductions presented here vary in generality from *MAP assignment*, *marginals*, and *full-joint estimation* to *pure-strategy Nash equilibria (PSNE)*, *mixed-strategy Nash equilibria (MSNE)*, and *correlated equilibria (CE)*, respectively. We briefly discuss a connection between Papadimitriou’s algorithm, as well as Jiang and Leyton-Brown’s, and the work of Jaakkola and Jordan [14] on variational approximations to the problem of probabilistic inference in MRFs via mean-field mixtures. The paper also includes a discussion on the connections to previous work in computer vision on the problem of relaxation labeling, and work on game-theoretic approaches to (Bayesian) statistical estimation. We then present an alternative approach based on a more global view of the problem, in contrast to the more local approach of the formulations mentioned above. More specifically, we formulate the inference problem using a two-player potential game, inspired by the work on *tree reweighed (TRW) message-passing* [15]. We propose a special type of sequential, “hybrid” standard and stochastic fictitious play algorithm for belief inference.

Section 4 reports on our experimental evaluation. We compare our proposed algorithms to the popular, most commonly used, standard, and easily implementable approximation techniques in use today.

Section 5 discusses future work and suggests new opportunities for other potential research directions, beyond those already discussed in the main technical sections of the paper.

Section 6 concludes the paper with a summary of our contributions.

2 Preliminaries

This section introduces basic notation and concepts in graphical models and game theory used throughout the paper. It also includes brief statements on current state-of-the-art mathematical and computational results in the area.

Basic Notation. Denote by $x \equiv (x_1, x_2, \dots, x_n)$ an n -dimensional vector and by $x_{-i} \equiv (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ the same vector without component i . Similarly, for every set $S \subset [n] \equiv \{1, \dots, n\}$, denote by $x_S \equiv (x_i : i \in S)$ the (sub-)vector formed from x using only components in S , such that, letting $S^c \equiv [n] - S$ denote the complement of S , we can denote $x \equiv (x_S, x_{S^c}) \equiv (x_i, x_{-i})$ for every i . If A_1, \dots, A_n are sets, denote by $A \equiv \times_{i \in [n]} A_i$, $A_{-i} \equiv \times_{j \in [n] - \{i\}} A_j$ and $A_S \equiv \times_{j \in S} A_j$.

Graph Terminology and Notation. Let $G = (V, E)$ be an undirected graph, with finite set of n vertices or nodes $V = \{1, \dots, n\}$ and a set of (undirected) edges E . For each node i , let $\mathcal{N}(i) \equiv \{j \mid (i, j) \in E\}$ be the set of neighbors of i in G , *not including* i , and $N(i) \equiv \mathcal{N}(i) \cup \{i\}$ the set *including* i . A *clique* C of G is a set of nodes with the property that they are all mutually connected: for all $i, j \in C$, $(i, j) \in E$; in addition, C is *maximal* if there is no other node k outside C that is also connected to each node in C , i.e., for all $k \in V - C$, $(k, i) \notin E$ for some $i \in C$.

Another useful concept in the context of this paper is that of hypergraphs, which are generalizations of regular graphs. A *hypergraph graph* $\mathcal{G} = (V, \mathcal{E})$ is defined by a set of nodes V and a set of *hyperedges* $\mathcal{E} \subset 2^V$. We can think of the hyperedges as cliques in a regular graph. Indeed, the *primal graph* of the hypergraph is the graph induced by the node set V and where there is an edge between two nodes if they both belong to the same hyperedge; in other words, the primal graph is the graph induced by taking each hyperedge and forming cliques of nodes in a regular graph.

2.1 Probabilistic Graphical Models

Probabilistic graphical models are an elegant marriage of probability and graph theory that has had tremendous impact in the theory and practice of modern artificial intelligence, machine learning, and statistics. It has permitted effective modeling of large, structured high-dimensional complex systems found in the real world. The language of probabilistic graphical models allows us to capture the structure of complex interactions between individual entities in the system within a single model. The core component of the model is a graph in which each node i corresponds to a random variable

X_i and the edges express conditional independence assumptions about those random variables in the probabilistic system.

2.1.1 Markov Random Fields, Gibbs Distributions, and the Hammersley-Clifford Theorem

By definition, a joint probability distribution P is a *Markov random field (MRF)* with respect to (wrt) an undirected graph G if for all x , for every node i , $P(X_i = x_i \mid X_{-i} = x_{-i}) = P(X_i = x_i \mid X_{\mathcal{N}(i)} = x_{\mathcal{N}(i)})$. In that case, the neighbors/variables $X_{\mathcal{N}(i)}$ form the *Markov blanket* of node/variable X_i .

Also by definition, a joint distribution P is a *Gibbs distribution* wrt an undirected graph G if it can be expressed as $P(X = x) = \prod_{C \in \mathcal{C}} \Phi_C(x_C)$ for some functions Φ_C indexed by a clique $C \in \mathcal{C}$, the set of all (maximal) cliques in G , and mapping every possible value x_C that the random variables X_C associated with the nodes in C can take to a non-negative number.

We say that a joint probability distribution P is *positive* if it has full support (i.e., $P(x) > 0$ for all x).⁴

Theorem 1. (Hammersley-Clifford [16]) *Let P be a positive joint probability distribution. Then, P is an MRF with respect to G if and only if P is a Gibbs distribution with respect to G .*

In the context of the theorem, the functions Φ_C are positive, which allows us to define MRFs in terms of *local potential functions* $\{\phi_C\}$ over each clique C in the graph. Define the function $\Psi(x) \equiv \sum_{C \in \mathcal{C}} \phi_C(x_C)$. Let us refer to any function of this form as a *Gibbs potential* with respect to G . A more familiar expression of an MRF is $P(X = x) \propto \exp(\sum_{C \in \mathcal{C}} \phi_C(x_C)) = \exp(\Psi(x))$.

2.1.2 Some Inference-Related Problems in MRFs

One problem of interest in an MRF is to compute a *most likely assignment* $x^* \in \arg \max_x P(X = x) = \arg \max_x \sum_{C \in \mathcal{C}} \phi_C(x_C)$; that is, the most likely outcome with respect to the MRF P . Another problem is to compute the *individual marginal probabilities* $P(X_i = x_i) = \sum_{x_{-i}} P(X_i = x_i, X_{-i} = x_{-i}) \propto \sum_{x_{-i}} \exp(\sum_{C \in \mathcal{C}} \phi_C(x_C))$ for each variable X_i . A related problem is to compute the normalizing constant $Z = \sum_x \exp(\sum_{C \in \mathcal{C}} \phi_C(x_C))$ (also known as the *partition function* of the MRF).

Another set of problems concern so called “belief updating.” That is, computing information related to the *posterior probability distribution* P' having observed the outcome of some of the variables, also known as the *evidence*. For MRFs, this problem is computationally equivalent to that of computing *prior* marginal probabilities.

2.1.3 Brief Overview of Computational Results in Probabilistic Graphical Models

Both the exact and approximate versions of most inference-related problems in MRFs are in general intractable (e.g., NP-hard), although polynomial-time algorithms do exist for some special cases (see, e.g., Istrail [1], Wang et al. [2], and the references therein). The complexity of exact algorithms is usually characterized by structural properties of the graph, and the typical statement is that running times are polynomial only for graphs with bounded treewidth (see, e.g., Russell and Norvig [17] for more information). Several deterministic and randomized approximation approaches exist (see, e.g., Jordan et al. [18], Jaakkola [19], Geman and Geman [20]).

2.2 Game Theory

Game theory [21] provides a mathematical model of the stable behavior (or outcome) that may result from the interaction of rational individuals. This paper concentrates in *noncooperative* settings: individuals maximize their *own* utility, act *independently*, and do not have (direct) control over the behavior of others.⁵

⁴The positivity constraint is only necessary for the “only if” case proof of the theorem.

⁵Individual rationality here means that each player seeks to maximize their own utility. Also note that, while many parlor “win-lose”/zero-sum games involve competition, in general, *noncooperative* \neq *competitive*: each player just wants to do the best for himself, regardless of how useful or harmful his behavior is to others.

The concept of *equilibrium* is central to game theory. Roughly, an equilibrium in a noncooperative game is a point of strategic stance, where no individual player can gain by *unilaterally* deviating from the equilibrium behavior.

2.2.1 Games and their Representation

Let $V = [n]$ denote a finite set of n players in a game. For each player $i \in V$, let A_i denote the set of *actions* or *pure strategies* that i can play. Let $A \equiv \times_{i \in V} A_i$ denote the set of *joint actions*, $x \equiv (x_i, \dots, x_n) \in A$ denote a joint action, and x_i the individual action of player i in x . Denote by $x_{-i} \equiv (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ the joint action of all the players *except* i , such that $x \equiv (x_i, x_{-i})$. Let $M_i : A \rightarrow \mathbb{R}$ denote the *payoff/utility function* of player i . If the A_i 's are finite, then M_i is called the *payoff matrix* of player i . Games represented this way are called *normal-* or *strategic-form games*.

There are a variety of compact representations for large game inspired by probabilistic graphical models in AI and machine learning [22–26]. The results of this paper are presented in the context of the following generalization of *graphical games* [23], a simple but powerful model inspired by probabilistic graphical models such as MRFs previously defined by Ortiz [27].⁶

Definition 1. A *graphical multi-hypermatrix game (GMhG)* is defined by

- a *directed* graph $G = (V, E)$ in which there is a node $i \in V$ in G for each of the n players in the game (i.e., $|V| = n$), and the set of directed edges, or arcs, E defines a set of *neighbors* $\mathcal{N}(i) \equiv \{j \mid (j, i) \in E, i \neq j\}$ whose action affect the payoff function of i (i.e., j is a neighbor of i if and only if there is an arc from j to i); and
- for each player $i \in V$,
 - a set of actions A_i ,
 - a hypergraph where the vertex set is its (*inclusive*) *neighborhood* $N(i) \equiv \mathcal{N}(i) \cup \{i\}$ and the hyperedge set is a set of *cliques* of players $\mathcal{C}_i \subset 2^{N(i)}$, and
 - a set $\{M'_{i,C} : A_C \rightarrow \mathbb{R} \mid C \in \mathcal{C}_i\}$ of *local-clique payoff (hyper)matrices*.

The interpretation of a GMhG is that, for each player i , the *local* and *global payoff (hyper)matrices* $M'_i : A_{N(i)} \rightarrow \mathbb{R}$ and $M_i : A \rightarrow \mathbb{R}$ of i are (implicitly) defined as $M'_i(x_{N(i)}) \equiv \sum_{C \in \mathcal{C}_i} M'_{i,C}(x_C)$ and $M_i(x) \equiv M'_i(x_{N(i)})$, respectively.

Graphical potential games. Graphical potential games are special instances of GMhGs. They play a key role in establishing a stronger connection between probabilistic inference in MRFs and equilibria in games than previously noted. Ortiz [11] provides a characterization of graphical potential games, and discusses the implication of convergence of certain kinds of “playing” processes in games based on connections to the Gibbs sampler [20], via the Hammersley-Clifford Theorem [16, 28]. Yu and Berthod [29] (implicitly) used graphical potential games to establish an equivalence between *local maximum-a-posteriori* (MAP) inference in Markov random fields and Nash equilibria of the game, a topic revisited in Section 3.1.⁷

2.2.2 Equilibria as Solution Concepts

Equilibria are generally considered *the* solutions of games. Various notions of equilibria exist. A *pure strategy (Nash) equilibrium (PSNE)* of a game is a joint action x^* such that for all players i , and for all actions x_i , $M_i(x_i^*, x_{-i}^*) \geq M_i(x_i, x_{-i}^*)$. That is, no player can improve its payoff by *unilaterally* deviating from its prescribed equilibrium x_i^* , assuming the others stick to their actions x_{-i}^* . Some games, such as the extensively-studied Prisoner’s Dilemma, have PSNE; many others, such as “playground” Rock-Paper-Scissors, do not. This is problematic because it will not be possible to “solve” some games using PSNE.

⁶Connections have already been established between the different kinds of compact representations [26], which may facilitate extensions of ideas, frameworks, and results to those alternative models.

⁷In the interest of brevity, please see Ortiz [27] for a thorough discussion of GMhGs, including their compact representation size and connections to other classical classes of games in game theory.

A *mixed-strategy* of player i is a probability distribution Q_i over A_i such that $Q(x_i)$ is the probability that i chooses to play action x_i .⁸ A *joint mixed-strategy* is a joint probability distribution Q capturing the players behavior, such that $Q(x)$ is the probability that joint action x is played, or in other words, each player i plays action in component x_i of x . Because we are assuming that the players play *independently*, Q is a product distribution: $Q(x) = \prod_i Q_i(x_i)$. Denote by $Q_{-i}(x_{-i}) \equiv \prod_{j \neq i} Q_j(x_j)$ the joint mixed strategies of all the players except i . The *expected payoff* of a player i when some joint mixed-strategy Q is played is $\sum_x Q(x) M_i(x)$; abusing notation, denote it by $M_i(Q)$. The *conditional expected payoff* of a player i given that he plays action x_i is $\sum_{x_{-i}} Q_{-i}(x_{-i}) M_i(x_i, x_{-i})$; abusing notation again, denote it by $M_i(x_i, Q_{-i})$.

A *mixed-strategy Nash equilibrium (MSNE)* is a joint mixed-strategy Q^* that is a product distribution formed by the individual players mixed strategies Q_i^* such that, for all players i , and any other alternative mixed strategy Q'_i for his play, $M_i(Q_i^*, Q_{-i}^*) \geq M_i(Q'_i, Q_{-i}^*)$. Every game in normal-form has at least one such equilibrium [30, 31]. Thus, every game has an MSNE “solution.”

One relaxation of MSNE considers the case where the amount of gain each player can obtain from unilateral deviation is very small. This concept is particularly useful to study approximation versions of the computational problem. Given $\epsilon \geq 0$, an (*approximate*) ϵ -*Nash equilibrium (MSNE)* is defined as above, except that the expected gain condition becomes $M_i(Q_i^*, Q_{-i}^*) \geq M_i(Q'_i, Q_{-i}^*) - \epsilon$.

Several refinements and generalizations of MSNE have been proposed. One of the most interesting generalizations is that of a *correlated equilibrium (CE)* [32]. In contrast to MSNE, a CE can be a full joint distribution, and thus characterize more complex joint-action behavior by players. Formally, a *correlated equilibrium (CE)* is a joint probability distribution Q over A such that, for all players i , $x_i, x'_i \in A_i$, $x_i \neq x'_i$, and $Q(x_i) > 0$, $\sum_{x_{-i}} Q(x_{-i}|x_i) M_i(x_i, x_{-i}) \geq \sum_{x_{-i}} Q(x_{-i}|x_i) M_i(x'_i, x_{-i})$, where $Q(x_i) \equiv \sum_{x_{-i}} Q(x_i, x_{-i})$ is the (marginal) probability that player i will play x_i according to Q and $Q(x_{-i}|x_i) \equiv Q(x_i, x_{-i}) / \sum_{x'_i} Q(x'_i, x_{-i})$ is the conditional given x_i . An MSNE is CE that is a product distribution. An equivalent expression of the CE condition above is

$$\sum_{x_{-i}} Q(x_i, x_{-i}) M_i(x_i, x_{-i}) \geq \sum_{x_{-i}} Q(x_i, x_{-i}) M_i(x'_i, x_{-i}).$$

As was the case for MSNE, we can relax the condition of deviation to account for potential gains from small deviation. Given $\epsilon > 0$, adding the term “ $-\epsilon$ ” to the right-hand-side of the condition above defines an (*approximate*) ϵ -*CE*.⁹

CE have several conceptual and computational advantages over MSNE. For instance, all players may achieve better expected payoffs in a CE than those achievable in any MSNE;¹⁰ some “natural” forms of play are guaranteed to converge to the (set of) CE [12, 33–37]; and CE is consistent with a Bayesian framework [38], something not yet possible, and apparently unlikely for MSNE [39].

2.2.3 Brief Overview of Results in Computational Game Theory

There has been an explosion of computational results on different equilibrium concepts on a variety of game representations and settings since the beginning of this century. The following is a brief summary. We refer the reader to a book by Nisan et al. [40] for a (partial) introduction to this research area.

The problem for two-player *zero-sum* games, where the sum of the entries of both matrix is zero, and therefore only one matrix is needed to represent the game, can be solved in polynomial time: It is equivalent to linear programming [21, 41, 42]. After being open for over 50 years, the problems of the complexity of computing MSNE in games was finally settled recently, following a very rapid sequence of results in the last part of 2005 [43–47]: Computing MSNE is likely to be hard in the worst case (i.e., PPAD-complete [48]), even in games with only two players [46, 49–52]. The result of Fabrikant et al. [53] suggests that computing PSNE in succinctly representable games is also likely to be intractable in the worst case (i.e., PLS-complete [54]). A common statement is that

⁸Note that the sets of mixed strategies contain pure strategies, as we can always recover playing a pure strategy exclusively.

⁹Note that approximate CE is usually defined based on this unconditional version of the CE conditions [33].

¹⁰The distinction between installing a traffic light at an intersection and leaving the intersection without one is a real-world example of this.

computing MSNE, and in some cases even PSNE, with “special properties” is hard in the worst case [55–57]. Computing approximate MSNE is also thought to be hard in the worst case [51, 58].

Most current results for computing exact and approximate PSNE or MSNE in graphical games essentially mirror those for MRFs and constraint networks: polynomial time for bounded treewidth graph; intractable in general [23, 27, 56, 59]. This is unsurprising because they were mostly inspired by analogous versions in probabilistic graphical models and constraint networks in AI, and therefore share similar characteristics. Several heuristics exist for dealing with general graphs [59–61].

In contrast, there exist polynomial-time algorithms for computing CE, both for normal-form games (where the problem reduces to a simple linear feasibility problem) and even most succinctly-representable games known today [8, 9], including graphical games.

3 Equilibria and Inference

The line of work presented in this section is partly motivated by the following question: *Can we leverage advances in computational game theory for problems in the probabilistic graphical models community?* Establishing a strong bilateral connection between both problems may help us answer this question.

The literature on computing equilibria in games has skyrocketed since the beginning of this century. As we discover techniques developed early on within the game theory community, and as new results are generated from the extremely active computational game theory community, we may be able to adapt those techniques for solving games to the inference setting. If we can establish a strong bilateral connection between inference problems and the computation of equilibria, we may be able to relate algorithms in both areas and exchange previously unknown results in each.

3.1 Pure-Strategy Nash Equilibrium and Approximate MAP Inference

Consider an MRF P with respect to graph G and Gibbs potential Ψ defined by the set of potential functions $\{\phi_C\}$. For each node i , denote by $\mathcal{C}_i \subset \mathcal{C}$ the subset of cliques in G that include i . Note that the (inclusive) neighborhood of player i is given by $N(i) = \cup_{C \in \mathcal{C}_i} C$.

Define an *MRF-induced* GMhG, and more specifically, a (hyperedge-symmetric) hypergraphical game [8, 11], with the same graph G , and for each player i , hypergraph with hyperedges \mathcal{C}_i and local-clique payoff hypermatrices $M_{i,C}^l(x_C) \equiv \phi_C(x_C)$ for all $C \in \mathcal{C}_i$. A few observations about the game are in order.

Property 1. *The representation size of the MRF-induced game is the same as that of the MRF: not exponential in the largest neighborhood size, but the size of the largest clique in G .*

Property 2. *The MRF-induced game is a graphical potential game [11] with graph G and (Gibbs) potential function Ψ : i.e., for all i , x and x'_i ,*

$$\begin{aligned} M_i(x_i, x_{-i}) - M_i(x'_i, x_{-i}) &= M_i^l(x_i, x_{N(i)}) - M_i^l(x'_i, x_{N(i)}) \\ &= \sum_{C \in \mathcal{C}_i} \phi_C(x_i, x_{C-\{i\}}) - \sum_{C \in \mathcal{C}_i} \phi_C(x'_i, x_{C-\{i\}}) \\ &= \sum_{C \in \mathcal{C}_i} \phi_C(x_i, x_{C-\{i\}}) + \sum_{C' \in \mathcal{C}-\mathcal{C}_i} \phi_{C'}(x_{C'}) \\ &\quad - \sum_{C \in \mathcal{C}_i} \phi_C(x'_i, x_{C-\{i\}}) - \sum_{C' \in \mathcal{C}-\mathcal{C}_i} \phi_{C'}(x_{C'}) \\ &= \Psi(x_i, x_{-i}) - \Psi(x'_i, x_{-i}). \end{aligned}$$

Remark 1. Through the connection established by the last property, it is easy to see that *sequential* best-response dynamics is guaranteed to converge to a PSNE of the game in finite time, regardless of the initial play.¹¹ In fact, we can conclude that a joint-action x^* is a PSNE of the game if and

¹¹ Recall that *best-response dynamics* refers to the a process where at each time step, each player observes the action x_{-i} of others and takes an action that maximizes its payoff given that the others played x_{-i} . In this case, those dynamics would essentially be implementing an axis-parallel coordinate maximization over the

only if x^* is a local maxima or a critical point of the MRF P . Thus, the MRF-induced game, like *all* potential games [62], always has PSNE.¹²

Similarly, for any potential game, one can define a *game-induced MRF* using the potential function of the game whose set of local maxima (and critical points) corresponds exactly to the set of PSNE of the potential game. Through this connection we can show that solving the local-MAP problem in MRFs is PLS-complete in general [53].¹³

One question that comes to mind is whether one can say anything about the properties of the globally optimal assignment in the game-induced MRF and the payoff it supports for the players. Or whether it can be characterized by stronger notions of equilibria. For example, are *strong NE*, in which no *coalition* of players could obtain a Pareto dominated set of payoffs by unilaterally deviating, joint MAP assignments of the MRF? Or more generally, what characteristics can we assign to the MAP assignments of the game-induced MRF?

In short, we can use algorithms for PSNE as heuristics to compute locally optimal MAP assignments of P and *vice versa*.¹⁴

Remark 2. Daskalakis et al. [66] extended results in game theory characterizing the number of PSNE in normal-form games (see Stanford [67], Rinott and Scarsini [68], and the references therein) to graphical games, but now taking into consideration the network structure of the game. Information about the number of PSNE in games can provide additional insight on the structure of MRFs.

For example, one of the results of Daskalakis et al. [66] states that for graphs respecting certain expansion properties as the number of nodes/players increases, the number of PSNE of the graphical game will have a limiting distribution that is a Poisson with expected value 1. Also according to Daskalakis et al. [66], a similar behavior occurs for games with graphs generated according to the Erdős-Rényi model with sufficiently high average-degree (i.e., reasonably high connectivity). Thus, either the set of MRF-induced games has significantly low measure relative to the set of all possible randomly generated games (something that seems likely), or the number of local maxima (and critical points) of the MRF will have a similar distribution, and thus that number is expected to be low. The latter would suggest that local algorithms such as the max-product algorithm may be less likely to get stuck in local maxima (or critical points) of the MRF.

In addition, there have been several results stating that PSNE are unlikely to exist in many graphs, and that, when they do exist, they are not that many [66].¹⁵ MRF-induced games would in that sense represent a very rich class of *non-randomly generated* graphical games for which the results above do not hold.

3.2 Mixed-strategy Equilibria and Belief Inference

Going beyond PSNE and MAP estimation, this subsection begins to establish a stronger, and potentially more useful connection between probabilistic inference and more general concepts of equilibria in games.

Let S be a subset of the players (i.e., nodes in the graph) and denote by $Q_S(x_S) \equiv \sum_{x_{V-S}} Q(x)$ the (marginal) probability distribution of Q over possible joint actions of players in S . Consider the space of assignments for the MRF, which is guaranteed to converge to a local maxima (or critical points) of the MRF.

¹²This result should not be surprising given that other researchers have established a one-to-one relationship between the complexity class PLS [54], which characterizes local search problems, of which finding local maxima of the MRF is an instance, and (ordinal) potential games [53].

¹³A direct proof of this result follows from [63], and in particular, the result for Hopfield neural networks [64]. A Hopfield neural network can be seen as an MRF, and more specifically, and Ising model, when the weights on the edges are symmetric. Similarly, any Hopfield neural network can be seen as a polymatrix game [65]; when the weights are symmetric the network can be seen as a potential game (in particular, it is an instance of a *party affiliation game* [53]). Indeed, a stable configuration in an arbitrary Hopfield neural network is equivalent to a PSNE of a corresponding polymatrix game. (See Papadimitriou et al. [63] and Miller and Zucker [65] for the relevant references.)

¹⁴Note that algorithms for PSNE can in principle find critical points of P . In either case, algorithms such as the max-product version of belief propagation (BP) can only provide such local-optimum/critical-point convergence guarantees in general.

¹⁵In particular, the number of PSNE has a Poisson distribution with parameter 1.

condition for correlated equilibria (CE), which for the MRF-induced game we can express as, for all $i, x_i, x'_i \neq x_i$,

$$\sum_{x_{N(i)}} Q_{N(i)}(x_i, x_{N(i)}) \sum_{C \in \mathcal{C}_i} \phi_C(x_i, x_{C-\{i\}}) \geq \sum_{x_{N(i)}} Q_{N(i)}(x_i, x_{N(i)}) \sum_{C \in \mathcal{C}_i} \phi_C(x'_i, x_{C-\{i\}}).$$

Commuting the sums and simplifying we get the following equivalent condition:

$$\sum_{C \in \mathcal{C}_i} \sum_{x_{C-\{i\}}} Q(x_i, x_{C-\{i\}}) \phi_C(x_i, x_{C-\{i\}}) \geq \sum_{C \in \mathcal{C}_i} \sum_{x_{C-\{i\}}} Q(x_i, x_{C-\{i\}}) \phi_C(x'_i, x_{C-\{i\}}). \quad (1)$$

This simplification is important because it highlights that, modulo expected payoff equivalence, we only need distributions over the original cliques, *not* the induced neighborhoods/Markov blankets, to represent CE in this class of games (in contrast to Kakade et al. [7]); thus, we are able to maintain the size of the representation of the CE to be the same as that of the game.

As an alternative, we can use the fact that the MRF-induced game is a potential game and, via some definitions and algebraic manipulation, get the following sequence of equivalent conditions, which hold for all i, x_i and x'_i .

$$\begin{aligned} \sum_{x_{-i}} Q(x_i, x_{-i}) (M_i(x_i, x_{-i}) - M_i(x'_i, x_{-i})) &\geq 0 \\ \sum_{x_{-i}} Q(x_i, x_{-i}) (\Psi(x_i, x_{-i}) - \Psi(x'_i, x_{-i})) &\geq 0 \\ \sum_{x_{-i}} Q(x_i, x_{-i}) (\ln P(x_i, x_{-i}) - \ln P(x'_i, x_{-i})) &\geq 0 \end{aligned}$$

Rewriting the last expression, we get the following equivalent condition: for all i, x_i and x'_i ,

$$\sum_{x_{-i}} Q(x_i, x_{-i}) [-\ln P(x_i, x_{-i})] \leq \sum_{x_{-i}} Q(x_i, x_{-i}) [-\ln P(x'_i, x_{-i})]. \quad (2)$$

The following are some additional remarks on the implications of the last condition.¹⁶

Remark 3. First, it is useful to introduce the following notation. For any distribution Q' , let $H(Q', P) \equiv \sum_x Q'(x) [-\log_2 P(x)]$ be the *cross entropy* between probability distributions Q' and P , with respect to P .¹⁷ Denote by $Q_{-i}(x_{-i}) \equiv \sum_{x_i} Q(x_i, x_{-i})$ the marginal distribution of play over the joint-actions of all players *except* player i . Denote by $Q'_i Q_{-i}$ the joint distribution defined as $(Q'_i Q_{-i})(x) \equiv Q'_i(x_i) Q_{-i}(x_{-i})$ for all x .

Then, condition 2 implies the following sequence of conditions, which hold for all i .

$$\begin{aligned} \sum_x Q(x) [-\ln P(x)] &\leq \sum_{x_{-i}} Q_{-i}(x_{-i}) [-\ln P(x'_i, x_{-i})] \text{ for all } x'_i. \\ H(Q, P) &\leq \min_{x'_i} \sum_{x_{-i}} Q_{-i}(x_{-i}) [-\log_2 P(x'_i, x_{-i})] \\ &= \min_{Q'_i} \sum_x Q'_i(x_i) Q_{-i}(x_{-i}) [-\log_2 P(x_i, x_{-i})] \\ &= \min_{Q'_i} H(Q'_i Q_{-i}, P) \end{aligned}$$

Hence, any CE of the MRF-induced game is a kind of approximate local optimum (or critical point) of an approximation of the MRF based on a special type of cross entropy minimization.

The following property summarizes this remark.

Property 3. For any MRF P , any correlated equilibria Q of the game induced by P satisfies $H(Q, P) \leq \min_i \min_{Q'_i} H(Q'_i Q_{-i}, P)$.

Remark 4. Let us introduce some additional notation. For any joint distribution of play Q' , let $H(Q') \equiv \sum_x Q'(x) [-\log_2 Q'(x)]$ be its entropy. Similarly, for any player i , for any marginal/individual distribution of play Q'_i , let $H(Q'_i) \equiv \sum_{x_i} Q'_i(x_i) [-\log_2 Q'_i(x_i)]$ be its (marginal) entropy. For any distributions Q' , let $\text{KL}(Q' \parallel P) \equiv \sum_x Q'(x) \log_2(Q'(x)/P(x)) = H(Q', P) - H(Q')$ be the *Kullback-Leibler divergence* between Q' and P , with respect to Q' . Denote by $H(Q_{i|-i}) \equiv \sum_{x_i, x_{-i}} Q(x_i, x_{-i}) \log_2(Q(x_i, x_{-i})/Q_{-i}(x_{-i})) = H(Q_{-i}) - H(Q)$ the conditional entropy of the individual play of player i given the joint play of all the players except i , with respect to Q .

¹⁶In what follows, we refer to concepts from information theory in the discussion, such as (Shannon's) entropy, cross entropy, and relative entropy (also known as Kullback-Leibler divergence). We refer the reader to Cover and Thomas [69] for a textbook introduction to those concepts.

¹⁷That is, (a lower bound on) the average number of bits required to transmit "messages/events" generated according to Q but encoded using a scheme based on P .

Then, we can express the condition 2 as the following equivalent conditions, which hold for all i .

$$\text{KL}(Q \parallel P) + H(Q) \leq \min_{Q'_i} \text{KL}(Q'_i Q_{-i} \parallel P) + H(Q'_i Q_{-i}) .$$

$$\text{KL}(Q \parallel P) + H(Q_{i|-i}) \leq \min_{Q'_i} \text{KL}(Q'_i Q_{-i} \parallel P) + H(Q'_i) .$$

Hence, any CE of a MRF-induced game is a kind of approximate local optimum (or critical point) of a special kind of variational approximation of the MRF. The following property summarizes this remark.

Property 4. For any MRF P , any correlated equilibria Q of the game induced by P satisfies $\text{KL}(Q \parallel P) \leq \min_i [\min_{Q'_i} \text{KL}(Q'_i Q_{-i} \parallel P) + H(Q'_i)] - H(Q_{i|-i})$.

Note that the last property implies that the approximation Q satisfies the local condition $\text{KL}(Q \parallel P) \leq \min_i \min_{Q'_i} \text{KL}(Q'_i Q_{-i} \parallel P) + \log_2 |\Omega_i|$.

Before continuing exploring connections to CE, it is instructive to first consider MSNE.

3.2.1 Mixed-strategy Nash Equilibria and Mean-Field Approximations

In the special case of MSNE, the joint mixed strategy $Q(x) = \prod_i Q_i(x_i)$ is a product distribution. Denote by $Q_{-i}^\times(x_{-i}) \equiv \prod_{j \neq i} Q_j(x_j) = \sum_{x_i} Q(x)$ the (marginal) joint action of play over all the players except i , and denote by $(Q'_i Q_{-i}^\times)$ the probability distribution defined such that the probability of x is $(Q'_i Q_{-i}^\times)(x) \equiv Q'_i(x_i) Q_{-i}^\times(x_{-i})$.

In this special case, the equilibrium conditions imply the following conditions, which hold for all i : for all x_i such that $Q_i(x_i) > 0$,

$$\sum_{x_{-i}} Q_i(x_i) Q_{-i}^\times(x_{-i}) [-\ln P(x_i, x_{-i})] = \min_{x'_i} \sum_{x_{-i}} Q_i(x_i) Q_{-i}^\times(x_{-i}) [-\ln P(x'_i, x_{-i})] .$$

This implies that

$$\begin{aligned} \sum_{x_i \text{ s.t. } Q_i(x_i) > 0} \sum_{x_{-i}} Q_i(x_i) Q_{-i}^\times(x_{-i}) [-\ln P(x_i, x_{-i})] = \\ \left(\sum_{x_i \text{ s.t. } Q_i(x_i) > 0} Q_i(x_i) \right) \min_{x'_i} \sum_{x_{-i}} Q_{-i}^\times(x_{-i}) [-\ln P(x'_i, x_{-i})] . \end{aligned}$$

The last condition is equivalent to

$$\sum_{x_i} \sum_{x_{-i}} Q_i(x_i) Q_{-i}^\times(x_{-i}) [-\ln P(x_i, x_{-i})] = \min_{x'_i} \sum_{x_{-i}} Q_{-i}^\times(x_{-i}) [-\ln P(x'_i, x_{-i})] ,$$

which, in turn, we can express as $H(Q, P) = \min_{Q'_i} H(Q'_i Q^\times, P)$. The last expression is also equivalent to

$$\text{KL}(Q \parallel P) + H(Q_i) = \min_{Q'_i} \text{KL}(Q'_i Q_{-i}^\times \parallel P) + H(Q'_i) .$$

Hence, a NE Q of the game is almost a locally optimal mean-field approximation, except for the extra entropic term. In summary, for MSNE we have the following tighter condition than for arbitrary CE.

Property 5. For any MRF P , any MSNE Q of the game induced by P satisfies $\text{KL}(Q \parallel P) = [\min_{Q'_i} \text{KL}(Q'_i Q_{-i}^\times \parallel P) + H(Q'_i)] - H(Q_i)$, for all i .

Note that the last property implies that the mean-field approximation Q satisfies the local condition $\text{KL}(Q \parallel P) \leq \min_{Q'_i} \text{KL}(Q'_i Q_{-i}^\times \parallel P) + \log_2 |\Omega_i|$ for all i .

One possible way to address the issue of the extra entropic term is to consider instead the *MRF-induced infinite game*, where each player i has the (continuous) utility function¹⁸

$$\widetilde{M}'_i(Q_i, Q_{\mathcal{N}(i)}) \equiv \sum_{x_i} \sum_{x_{\mathcal{N}(i)}} \left[Q_i(x_i) \prod_{j \in \mathcal{N}(i)} Q_j(x_j) \right] M'_i(x_i, x_{\mathcal{N}(i)}) + H(Q_i)$$

¹⁸In an *infinite game* the sets of actions or pure strategies are uncountable. Existence of equilibria holds under reasonable conditions (i.e., each set of actions is a nonempty compact convex subset of Euclidean space, and each player utility is continuous and quasi-concave in the player's action), all of which are satisfied by the MRF-induced infinite game considered here. (See Fudenberg and Tirole [70] for more information.)

and wants to maximize over its mixed-strategy Q_i given the other player mixed-strategies Q_j for all $j \neq i$.

Property 6. *The MRF-induced infinite game defined above is an infinite Gibbs potential game with the same graph G and the following potential over the set of individual (product) mixed strategies*

$$\Psi(Q) = \sum_{C \in \mathcal{C}} \sum_{x_C} \left[\prod_{j \in C} Q_j(x_j) \right] \phi_C(x_C) + H(Q) = -KL(Q \parallel P) + Z$$

where Z is the normalizing constant for P . From this we can derive that the individual player mixed-strategies $\{Q_i\}$ are a “pure strategy” equilibrium of the infinite game if and only if

$$KL(Q \parallel P) = \min_{Q'_i} KL(Q'_i Q_{-i}^\times \parallel P).$$

Or, in other words, if Q is a PSNE of the infinite game, then Q is also a local optimum (or critical point) of the mean-field approximation of P .

Remark 5. The local payoff function defined above for the infinite game also has connections to the game theory literature on *learning in games* [12]. This area studies properties of processes by which players “learn” how to play in (usually repeated) games; specially properties related to the existence of convergence of the learning (or playing) dynamics to equilibria. In particular, the local payoff function is similar to that used by *logistic fictitious play*, a special version of a “learning” process called *smooth fictitious play*. The difference is that the last entropy term involving the individual player’s mixed strategy has a regularization-type factor $\lambda > 0$ such that players play strict best-response as $\lambda \rightarrow 0$. In addition, logistic fictitious play is an instance of a learning process that, if followed by a player, achieves so called approximate *universal consistency* (i.e., roughly, in the limit of infinite play, the average of the payoffs obtained by the player will be close to the best obtained overall during repeated play, *regardless of how the other players behave*), for appropriate values of λ depending on the desired approximation level.

Indeed, it is not hard to see that in fact the best-response mixed-strategy Q_i of player i to the mixed strategies $Q_{\mathcal{N}(i)}$ of their neighbors is

$$\begin{aligned} Q_i(x_i) &\propto \exp(\sum_{x_{\mathcal{N}(i)}} \left[\prod_{j \in \mathcal{N}(i)} Q_j(x_j) \right] M'_i(x_i, x_{\mathcal{N}(i)})) \\ &= \exp(\sum_{C \in \mathcal{C}_i, C \neq \{i\}} \left[\prod_{j \in C - \{i\}} Q_j(x_j) \right] \phi_C(x_i, x_{C - \{i\}})) . \end{aligned}$$

Hence, running *sequential* best-response dynamics in the MRF-induced infinite game is equivalent to finding a variational mean-field approximation via recursive updating of the first derivative conditions.¹⁹ The process will then be equivalent to minimizing the function $F(Q) \equiv KL(Q \parallel P)$ by axis-parallel updates. The resulting sequence of distributions/mixed-strategies monotonically decreases the value of F and is guaranteed to converge to a local optimum or a critical point of F . Hence, the corresponding learning process is guaranteed to converge to a PSNE of the *infinite* game, which is in turn *approximate* MSNE of the *original* game. But this is not surprising in retrospect, given the last property (Property 6). That property essentially states a broader property of *all* potential games: they are isomorphic to so called *games with identical interests* [62], which are games where every player has exactly the same payoff function.

Remark 6. The previous discussion suggests that we could use appropriately-modified versions of algorithms for MSNE, such as **NashProp** [61], as heuristics to obtain a mean-field approximation of the true marginals.

Going in the opposite direction, the discussion above also suggests that, by treating any (graphical) potential game as an MRF, for any fixed $\lambda > 0$, logistic fictitious play in any potential game converges to an approximate $(\lambda / \min_i |A_i|)$ -MSNE of the potential game. Indeed, there has been recent work in this direction, which explores the connection between learning in games and mean-field approximations in machine learning [71]. That work proposes new algorithms based on fictitious play for simple mean-field approximation applied to statistical (Bayesian) estimation.

The game-induced MRF is a λ -temperature Gibbs measure. As we take $\lambda \rightarrow 0$, we get the limiting 0-temperature Gibbs measure which is a probability distribution over the set of global maxima of

¹⁹In particular, the process is called a *Cournot adjustment with lock-in* in the literature on learning in games [12].

the potential function of the game, and 0 probability everywhere (i.e., the support of the limiting distribution is the set of joint-actions that maximize the potential function). The support of the 0-temperature Gibbs measure is a subset of the “globally optimal” PSNE of the potential game. But there might be other equilibria corresponding to local optima (or critical points) of the potential function.

Are there other connections between the Nash equilibria of the game and the support of the limiting distribution?

3.2.2 Correlated Equilibria and Higher-order Variational Approximations

Kakade et al. [7] designed polynomial-time algorithms based on linear programming for computing CE in standard graphical games with tree graphs. The approach and polynomial-time results extend to graphical games with bounded-tree-width graphs and graphical polymatrix games with tree graphs. Ortiz et al. [72] (see also Ortiz et al. [73]) proposed the principle of maximum entropy (MaxEnt) for equilibrium selection of CE in graphical games. They studied several properties of the MaxEnt CE, designed a monotonically increasing algorithm to compute it, and discussed a learning-dynamics view of the algorithm. Kamisetty et al. [74] employed advances in approximate inference methods to propose approximation algorithms to compute CE. In all of those cases, the general approach is to use ideas from probabilistic graphical models to design algorithms to compute CE. The focus of this paper is the opposite direction: employing ideas from game theory to design algorithms for belief inference in probabilistic graphical models.

Property 4 suggests that we can use the CE for the MRF-induced game as a heuristic approximation to higher-order variational approximations. In fact, one would argue that in the context of inference, doing so is more desirable because, in principle, it can lead to better approximations that can capture more aspects of the joint distribution than a simple mean-field approximation would alone. For example, mean-field approximations are likely to be poor if the MRF is multi-modal. Motivated by this fact, Jaakkola and Jordan [14] suggest using mixture of product distributions to improve the simple variational mean-field approximation.

3.2.3 Some Computational Implications

But, consider the algorithms of Papadimitriou [8] or Jiang and Leyton-Brown [9] (see also Papadimitriou and Roughgarden [6] and Jiang and Leyton-Brown [10]), which we can use to compute a CE of the MRF-induced game in polynomial time. Such CE will be, by construction, also a (*polynomially-sized*) *mixture of product distributions*. (In the case of Jiang and Leyton-Brown’s algorithm it will be a mixture of a subset of the joint-action space, which is equivalent to a probability mass function over a *polynomially-sized* subset of the joint-action space; said differently, a mixture of product of indicator functions, each product corresponding to particular outcomes of the joint-action space.) Hence, the algorithms of Papadimitriou and Jiang and Leyton-Brown both provide a means to obtain a heuristic estimate of a local optimum (or critical point) of such a mixture *in polynomial time*. The result would not be exactly the same as that obtained by Jaakkola and Jordan [14] in general, because of the extra entropic term mentioned in the discussion earlier. *Can we find alternative versions of the payoff matrices, and/or alter Papadimitriou’s algorithm, so that the resulting correlated equilibria provides an exact answer to the approximate inference problem that uses mixtures of product distributions?* Regardless, at the very least one could use the resulting CE to initialize the technique of Jaakkola and Jordan [14] without specifying an *a priori* number of mixtures.

Having said that, both Papadimitriou’s and Jiang and Leyton-Brown’s algorithms make a polynomial number of calls to the ellipsoid-algorithm, or more specifically, its “oracle,” to obtain each of the product distributions whose mixture will form the output CE. It is known that the ellipsoid algorithm is slow in practice. Papadimitriou [8], Papadimitriou and Roughgarden [6], and Jiang and Leyton-Brown [9] leave open the design of more practical algorithms based on interior-point methods.

Finally, this connection also suggests that we can (in principle) use any learning algorithm that guarantees convergence to the set of CE (as described in the section on preliminaries on game theory where the concept was introduced) as a heuristic for approximate inference. Several so-called “no-regret” learning algorithms satisfy those conditions. Indeed, we use a variant of a simple version of a “no-regret” algorithm in our experiments. Viewed that way, such learning algorithms would be similar in spirit to stochastic simulation algorithms with a kind of “adaptivity” reminiscent of the

work on adaptive importance sampling (see, e.g., Cheng and Druzdzal [75], Ortiz and Kaelbling [76], Ortiz [77], and the references therein). Establishing a possible stronger connection between learning in games, CE, and probabilistic inference seems like a promising direction for future research. In fact, as previously mentioned (at the end of Remark 5), there has already been some recent work in this direction, but specifically for MSNE and mean-field approximations [71].

Later in this paper, we present the results of an experimental evaluation of the performance of a simple no-regret learning algorithm in computational game theory in the context of probabilistic inference. We delay the details until the Experiments section.

3.3 Other Previous and Related Work

Earlier work on the so called “relaxation labeling” problem in AI and computer vision [78, 79] has established connections to polymatrix games [80] (see also Hummel and Zucker [81], although the connection had yet to be recognized at that time). That work also establishes connections to inference in Hopfield networks, dynamical systems, and polymatrix games [79, 82]. A reduction of MAP to PSNE in what we call here a GMhG was introduced by Yu and Berthod [29] in the same context (see also Berthod et al. [83]); although they concentrate on pairwise potentials, which reduce to polymatrix games in this context. Because, in addition, the ultimate goal in MAP inference is to obtain a *global* optimum configuration, Yu and Berthod [29] proposed a Metropolis-Hastings-style algorithm in an attempt to avoid local minima. Their algorithm is similar to simulated annealing algorithms used for solving satisfiability problems, and other local methods such as WalkSAT [84] (see, e.g., Russell and Norvig [17] for more information). The algorithm can also be seen as a kind of learning-in-games scheme [12] based on best-response with random exploration (or “trembling hand” best response). That is, at every round, some best-response is taken with some probability, otherwise the previous response is replayed. Zucker [82] presents a modern account of that work. The connection to potential games, and all its well-known properties (e.g., convergence of best-response dynamics) does not seem to have been recognized within that literature. Also, none of the work makes connections to higher-order (i.e., beyond mean-field) inference approximation techniques or the game-theoretic notion of CE.

3.4 Approximate Fictitious Play in a Two-player Potential Game for Belief Inference in Ising Models

This section presents a game-theoretic fictitious-play approach to estimation of node-marginal probabilities in MRFs. The approach this time is more global in terms of how we use the whole joint-distribution for the estimation of individual marginal probabilities. The inspiration for the approach presented here falls from the work of Wainwright et al. [15]. The section concentrates on Ising models, an important, special MRF instance from statistical physics with its own interesting history.

Definition 2. An *Ising model* wrt an undirected graph $G = (V, E)$ is an MRF wrt G such that

$$\mathbf{P}_\theta(x) \propto \exp \left(\sum_{i \in V} b_i x_i + \sum_{(i,j) \in E} w_{i,j} x_i x_j \right)$$

where $\theta \equiv (\mathbf{b}, \mathbf{W})$ is the set of node biases b_i ’s and edge-weights $w_{i,j}$ ’s, which are the parameters defining the joint distribution \mathbf{P}_θ over $\{-1, +1\}^n$.

It is fair to say that interest on more general classes of MRFs originates from the special class of Ising models. It is also fair to say that, because of the relative simplicity and importance of Ising models for problems in statistical physics, as well as to other ML and AI applications areas such as computer vision and NLP, Ising models have become the most common platforms in which to empirically study approximation algorithms for arbitrary MRFs. In short, simplicity of presentation and empirical evaluation guide the focus of Ising models in this section: Generalizations to arbitrary MRFs are straightforward but cumbersome to present. Hence, in this manuscript, we omit the details of such generalizations.

As an outline, the current section begins with an algorithmic instantiation of the iterative approach. The exact instantiation depends on whether we are using CE or MSNE as the solution concept.

The section then follows with an informal discussion of the game-theoretic foundations of the general framework behind the approach, and a discussion of immediate implications to computational properties and potential convergence.

Denote by \mathbb{T}_G the set of all spanning trees of connected (undirected) graph $G = (V, E)$ that are maximal with respect to E (i.e., does not contain any spanning forests). If spanning tree $T \in \mathbb{T}_G$, we denote by $E(T) \subset E$ the set of edges of T .

Initialize $x^{(1)} \leftarrow \text{Uniform}(\{-1, +1\}^n)$, and for each $(i, j) \in E$, $\hat{\mu}_{(i,j)}^{(1)} \leftarrow x_i^{(1)} x_j^{(1)}$. At each iteration $l = 1, 2, \dots, m$,

- 1: $T^{(l)} \leftarrow \text{Uniform}\left(\arg \max_{T \in \mathbb{T}_G} \sum_{(i,j) \in E} \mathbb{1}[(i,j) \in E(T)] w_{ij} \hat{\mu}_{(i,j)}^{(l)}\right)$
- 2: $s_l \leftarrow \text{Uniform}(\{1, \dots, l\})$
- 3: $x^{(l+1)} \leftarrow \text{Uniform}\left(\arg \max_{x \in \{-1, +1\}^n} \sum_{i \in V} b_i x_i + \sum_{(i,j) \in E(T^{(s_l)})} w_{ij} x_i x_j\right)$
- 4: **for all** $(i, j) \in E$ **do**
- 5: $v_{(i,j)}^{(l+1)} \leftarrow x_i^{(l+1)} x_j^{(l+1)} \times \begin{cases} 1, & \text{if MSNE,} \\ \mathbb{1}[(i,j) \in E(T^{(s_l)})], & \text{if CE} \end{cases}$
- 6: $\hat{\mu}_{(i,j)}^{(l+1)} \leftarrow \frac{l \hat{\mu}_{(i,j)}^{(l)} + v_{(i,j)}^{(l+1)}}{l+1}$
- 7: **end for**

For each Ising-model's random-variable index $i = 1, \dots, n$, set $p_i^{(m+1)} = \frac{1}{m+1} \sum_{l=1}^{m+1} \mathbb{1}[x_i^{(l)} = 1]$ as the estimate of the exact Ising-model's marginal probability $p_i \equiv \mathbf{P}(X_i = 1)$.

Within the literature on probabilistic graphical models, Hamze and de Freitas [85] propose an MCMC approach based on sampling non-overlapping trees. While our approach has a sampling flavor, its exact connection to MCMC is unclear at best. Also, the spanning trees that our algorithm generates may overlap.

The following discussion connects the algorithm above to an approximate version of fictitious play from the literature on learning in games in game theory. For the most part, we omit discussions to approximate variational inference in this manuscript, except to say that TRW message-passing [15] is the inspiration behind our proposed algorithm above.

The game implicit in the heuristic algorithm above is a two-player potential game between a “joint-assignment” (JA) player and a “spanning-tree” (ST) player. The potential function is $\Psi_{X,T}(x, T) = \sum_{i \in V} b_i x_i + \sum_{(i,j) \in E} \mathbb{1}[(i,j) \in E(T)] w_{ij} x_i x_j$. The payoff functions M_X and M_T of the JA player and the ST player, respectively, are identical and equal the potential function $\Psi_{X,T}(x, T)$: formally, $M_X(x, T) = M_T(x, T) = \Psi_{X,T}(x, T)$. Note that the payoff function of the ST player is *strategically equivalent* to the function $\sum_{(i,j) \in E} \mathbb{1}[(i,j) \in E(T)] w_{ij} x_i x_j$.

Technically, this is a game with identical payoffs, which are known to have what Monderer and Shapley [86] called the *fictitious play property*: the empirical play of fictitious play is guaranteed to converge to an MSNE of the game. While determining a best-response for the ST player is easy (i.e., using an algorithm for computing maximal spanning tree such as Kruskal's, as we do in our implementation for the experiments), unfortunately the same is in general not possible for the JA player, whose best-response is as hard as computing a MAP assignment of another Ising model with the same graph and (generally non-zero) bias/node parameters, but a slightly different set of edge-weights.²⁰

One approach to deal with the problem of obtaining a best-response from the JA player is to draw one tree uniformly at random from the empirical distribution and find a best-response to that tree. Such an approach is equivalent to a type of smooth best-response. If both players were to do the same, *simultaneously*, the result is a stochastic version of fictitious play or *stochastic fictitious play*

²⁰ As mentioned earlier, there are some instances for which this computation is actually possible in polynomial time. In fact, this would have been possible for the type of Ising models with planar two-dimensional grid graph, also known as a “square lattices,” we used in the experiments, if we would have chosen those models to have zero biases, or the edge-weights had some special characteristics. Unfortunately, there is no guarantee that the specific Ising models randomly drawn would satisfy those conditions in general. As we discuss shortly, we settle for a simple computation of the best-response for the JA player using stochastic fictitious play [12].

for short [12]. The empirical distribution of play of stochastic fictitious play in a game with identical payoffs, or what’s strategically equivalent, any potential game, also converges to an MSNE of the game [87]. In our case, however, we really have a type of “hybrid” sequential-version, where the ST player is always behaving as in standard fictitious play, while the JA player is behaving according to a stochastic fictitious play.

In addition, as an alternative to the best-response computation for player JA, we might want to add an entropic (preference) function of the mixed-strategy to the JA player as an additional term in JA’s payoff, so that the result is really a “smooth” best-response, or more specifically in this case a *smooth stochastic fictitious play* [12]. Such an addition would make the connection to variational inference more evident, and would allow us to develop more direct bounds on the quality of the variational result. The main problem is that we do not know of any study of such hybrids within game theory. In addition, most instances of fictitious play assume *simultaneous* moves. Numerical instability is another problem we found in practice when using such smooth best-response. Even in instances where that was not a problem, the performance was indistinguishable, in a formal statistical sense, from the version of the algorithm that we propose above.

In the context of belief inference, we believe it actually makes more sense to have a so called “sequential” play, where players trade moves: the JA player starts by choosing some action (i.e., full, joint assignments to the random variables), the ST player best-responds to that action, then the JA player best-responds to the ST player’s action, continuing in that way, such that at each round, each player is best-responding to the *empirical* distribution of play ²¹ up to the time the player makes a move (i.e., draws an action). While this type of sequential process often helps to stabilize the dynamics and improve the likelihood of convergence, it seems that such sequential processes have received considerably less attention than their simultaneous-move counterpart within the game-theory community.

We conjecture, however, that the type of fictitious play process defined above in fact converges. We believe that the proof follows from combining results from standard and stochastic fictitious play for games with identical payoffs, which are (strategically equivalent) instances of potential games [86, 87]. The derivation is complex and not trivial, involving key mathematical concepts from the literature in stochastic approximation. Delving into such level of complexity not only goes beyond the scope of this paper, but more importantly, doing so distracts attention from the paper’s main focus: to provide a general, broad illustration of how ideas and results from game theory may be useful in providing alternative, effective, and practical approaches to hard belief-inference problems in probabilistic graphical models. Thus, we leave the formal proof of our conjecture as future work.

As a last point, it is important to understand and keep in mind that, as it is well-known, in the context of potential games, while *sequential best-reply* converges to a PSNE (i.e., a joint assignment), *fictitious play* can converge to an MSNE of the game. ²² Monderer and Shapley [86] provide an example in a 2-player 2-action normal-form (coordination) game with identical payoffs. Said differently, the resulting empirical distribution of play for the JA player may be to what Monderer and Shapley [86] themselves call a “purely mixed strategy” (i.e., every action is played with positive probability; or said differently, the corresponding probability mass function has full support over the action set of the player). ²³ In the context of belief inference, the resulting mixed-strategy would correspond to an (approximate) marginal distribution, not a particular joint-assignment. Hence, in the context of belief inference, the convergence of the procedure above may not have to be to a single (possibly local) optimum of the potential function $\Psi_{X,\mathcal{T}}$: in principle, convergence could be to a (non-deterministic) *mixture* over joint-assignments. In fact, this is what we observe in our experiments, albeit after only a finite number of iterations. A thorough understanding of the convergence properties observed in practice requires considerably more experimental work than is reasonable within the context and purpose of the work described in this manuscript.

²¹In game theory, this is also known as the *belief* distribution of play each player has about the others’ future mixed-strategy based on previously observed play.

²²Recall that in fictitious play, each player uses the *empirical distribution of play* as an estimate or belief of how the other player would behave in the future, *not just* the other player’s *last action* as in sequential best-reply.

²³Other names used in game theory are *totally mixed strategy* or *mixed strategy with full support*.

4 Experiments

In this section we present the results of synthetic experiments on the performance of the game-theoretic-inspired heuristics we propose in this paper for approximate belief inference in MRFs. Our algorithms have very simple implementations. We also compare them with the most popular approximation algorithms and heuristics, with equally simple implementations, proposed in the literature on probabilistic graphical models.

4.1 Experimental Design

The experimental design in terms of the class of Ising models is as in Domke and Liu [88]. We consider Ising models with $d \times d$ simple grid graphs, which are planar (i.e., no “wrap around” edges, such that each of the four corner nodes have exactly two neighboring nodes, any other non-internal node has exactly three neighbors, while the rest, i.e., all internal nodes, have exactly four neighbors). Hence, the number of variables/nodes is $n = d^2$. Also we did not consider edge-weights magnitude parameters 1.0 or 1.5, because it is really hard to beat a Gibbs sampler for maximum weight magnitudes smaller than 2.0, relative to the bias parameters b_i ’s being in the real-valued interval $[-1, 1]$. The reason for this might be that, because, as stated in Domke and Liu [88], the mixing rate of a Gibbs sampler in such models grows roughly exponential with the magnitude, the induced Markov chain mixes pretty fast for such cases; thus convergence is quick. For each value $w \in \{2, 3, 4\}$, we generated random Ising models with edge-weights $w_{ij} \sim \text{Uniform}([-w, w])$ or $w_{ij} \sim \text{Uniform}([0, w])$ for the “mixed” or “attractive” case, respectively, for each $(i, j) \in E$, i.i.d., and node biases $b_i \sim \text{Uniform}([-1, +1])$, also i.i.d. for all i , and independent of the edge-weights.

One exception on the class of Ising models used for evaluation is a class we use with edge-weights with constant *magnitude* (i.e., $w = \max_{(i,j) \in E} |w_{ij}|$), but in which we vary the probability q of attractive edge-weights; that is, given a probability q , the sign of the edge-weight are i.i.d. random variables in which the sign is positive with probability q , and negative with probability $1 - q$.²⁴ We propose this class of Ising models for future evaluations of approximate belief inference techniques. For evaluation using this class, we consider $w \in \{2.0, 2.5, 3.0, 4.0\}$. For each q , we randomly generated 50 Ising models as samples for $w = 4$, and 5 samples for each $w \neq 4$.

Note that despite the graphs being planar, the bias parameter is non-zero in general, so that the known polynomial-time exact algorithms for planar graphs do not technically apply.

Here, we consider a simple *no-regret algorithm*, which we denote as “nr” from now on. The algorithm guarantees to converge to (the *set* of approximate) CE [33]. While there are several such no-regret algorithms in the literature with the same convergence guarantees, we leave their evaluation for future work. Each iteration of nr takes roughly the same amount of time as that for Gibbs sampling. We set the number of iterations of the nr algorithm to 10^5 for the standard experimental setup, and to 10^6 for our proposed new evaluation setting. Our exact implementation is a natural adaptation we believe is more amenable to the belief-inference setting. In particular, we evaluate a version in which we update the mixed-strategy each player uses to draw an action at every iteration t as follows. For each player, (1) we set the probability of switching the player’s last action being equal to the empirical regret, or 0 if the empirical regret is negative; and (2) we set the player’s probability of playing action $+1$ by “damping” the currently suggested probability of playing $+1$, $p_t(1)$, for the corresponding player by the original algorithm:²⁵ that is, we use the update $0.99 \times p_t(1) + 0.01 \times (0.5)$. We also use 10^5 iterations. Also, we only present results for the sequential, “semi-stochastic” fictitious play we discuss in Section 3.4, for the case of CE only, which we denote as “fp (ce)” from now on. We set the number of iterations $m = 15$.²⁶ Finally, the

²⁴The weight of each edge $(i, j) \in E$ is a random variable of the form $W_{ij} = (2S_{ij} - 1)w$, where the $S_{ij} \sim \text{Bernoulli}(q)$, i.i.d., and w is a positive constant.

²⁵The algorithm basis its suggested probability solely on the positively-truncated empirical regret.

²⁶That number of iterations is relatively low, but given that our implementation is in Octave, setting $m = 15$ is roughly the number of iterations for which the amount is roughly the same as that for our C implementation of TRW, as described in Wainwright et al. [15], but without optimizing for the parameters ρ_{ij} ’s, which we set to a constant $= 0.55$ for all edges $(i, j) \in E$. Clearly this is an unfair comparison for fp (ce). The optimization of ρ_{ij} ’s involves performing a maximum spanning tree computation at each iteration until convergence, and this operation follows each TRW message-passing with fixed ρ_{ij} ’s. While such an optimization is tractable, and optimizing for the ρ_{ij} ’s does seem to improve the upper-bounds on the log-partition function, it is not clear

results for the MSNE-instantiation of the fictitious play algorithm we propose are quite similar to those for fp (ce), at least for $m = 15$; thus, we omit those results in the interest of keeping the plots less “crowded” and thus easier to interpret.

We compare the simple nr algorithm and our proposed fp (ce) to (1) standard mean-field approximation (mf), with sequential/axis-parallel updates; (2) standard belief propagation (bp), with simultaneous updates; (3) TRW (trw); and (4) the Gibbs sampler (gs). In the next paragraph, we provide more detail on the specifics of the implementations of methods (1–4).

As baseline (bl), we use the simplest possible estimator from the perspective of average marginal-error to measure quality: always use 0.5 as the estimate of the exact marginal distribution of each variable. Certainly, one would expect that for an algorithm to be competitive, its performance should be better than bl. As we soon discuss, our experimental results suggest that this is not always the case; that is, several standard methods, including some of the ones proposed here and even state-of-the-art such as TRW, do not satisfy that condition for “hard” cases. We evaluate mean field (mf) using sequential/axis-parallel updates, stopping if the maximum absolute difference in probability values between iterations is $\leq 10^{-5}$, and using a maximum number of iterations = 10^6 . For belief propagation (bp) we use simultaneous updates, and “smooth” the update based on the average of the current value and the new value in order to “dampen” or at least try to prevent oscillations and improve the likelihood of convergence,²⁷ stopping if the maximum absolute difference in probability values between iterations is $\leq 10^{-7}$, and a maximum number of iterations = 10^5 . For tree reweighed message-passing (trw), we use a constant parameter $\rho = 0.55$ for all corresponding edge-appearance-probability parameters ρ_{ij} ’s [15], along with a smooth update and the same stopping criterion as for bp. For the Gibbs sampler, we use 10^6 iterations.

4.2 Experimental Results and Discussion

Fig. 1 summarizes our results for the most common classes of Ising models considered in the experimental evaluation of approximation algorithms and heuristic for belief inference in the literature as described above. We perform hypothesis testing for the result in these classes of Ising models using paired z-tests on the individual (i.e., not joint) differences, each with p-value 0.05. Hence, all the statements are statistically significant with respect to such hypothesis tests. Note that there is no globally best approximation technique overall for these classes.

“Mixed” case (Left plot, Fig. 1). Clearly, gs is best for all w in this case. Among the other approximation algorithms, we observe the following

1. fp (ce) is best and better than bp for $w = 4$, indistinguishable from bp for $w = 3$, and worst than bp for $w = 2$ where bp is best.
2. fp (ce) is consistently better than trw.
3. trw is worst than bp for $w < 4$, but better than bp for $w = 4$.
4. All methods, except for mf and nr, are consistently better than bl; mf and nr are consistently worst than bl, except for $w = 2$ where mf is indistinguishable from bl.
5. mf and nr are indistinguishable, except for $w = 4$ where nr is better than mf.

“Attractive” case (Right plot, Fig. 1). In this case, there is no clear overall best. We also observe the following.

1. trw is best among all methods except for $w = 2$ where gs is best, and trw is second best.
2. fp (ce) is better than all other methods, except trw, and gs for $w = 2$; and bp for $w < 4$ where fp (ce) is indistinguishable from bp.
3. mf, nr, bp, and gs are consistently indistinguishable from bl, and from each other; except for $w = 2$ where gs is best, of course.

from the experimental results in Wainwright et al. [15] that the improvement on the quality of the individual marginal estimates justify the extra work necessary for the optimization.

²⁷It is well known that bp may not converge in MRFs with loopy graph, such as the Ising model with grid graph we are using here for our experiments.

Fig. 2 summarizes our experimental results for a class of Ising models which appears to lead to “harder” Ising-model instances.²⁸ We perform hypothesis testing for the result in these classes of Ising models using two approaches depending on w . For $w = 4$, where we draw 50 models as samples for each q , we use appropriately modified paired z-tests on the individual (i.e., not joint) differences, each with p-value 0.05. We modify the calculation of the variances resulting from the average over the samples computed for each q . We do so because the distributional properties of the empirical mean/average for each q may differ. For $w < 4$, where we only draw 5 models as samples for each q , we use bootstrapped-based, individual, paired hypothesis-testing over each pair of aggregate differences between the methods for each of those values of w ; we use 100 bootstrap samples, and p-value 0.05. All the statements are statistically significant with respect to such hypothesis tests.

Aggregate results (Left plot, Fig. 2). The left-hand plot in Fig. 2 shows the aggregate results for this case. There is no clear overall best over all q . We also observe the following.

1. fp (ce) is best for $w = 4$, while being second best to gs for $w = 2$ and to trw for $w = 3$; and, for $w = 2.5$, tied for best with gs (i.e., indistinguishable from gs).
2. trw is consistently better than bp, mf, bl, and nr, except for $w = 2.5$ where trw is indistinguishable from bp; trw is consistently worst than fp (ce), except for $w = 3$ where trw is best, of course;
3. Only mf is worst than bl for $w \in \{2, 4\}$; mf is indistinguishable from bl for $w \in \{2.5, 3\}$; also, bp and nr are indistinguishable from bl, except for $w = 2.5$, where bp is better than bl.
4. bp is better than mf except for $w = 2$; bp is indistinguishable from nr for $w \in \{2, 3\}$, but bp is better than nr for $w \in \{2.5, 4\}$.
5. gs is consistently better than mf and nr; indistinguishable from trw except for $w = 3$ where trw is better, and, of course, for $w = 2$ where gs is tied for best with fp (ce).

Results for constant edge-weight magnitude $w = 4$ as a function of probability of attractive interaction q (Right plot, Fig. 2). The right-hand plot in Fig. 2 shows finer-grain results for this case. The results suggest that in fact such instances of Ising models tend to be harder in the sense that even state-of-the-art algorithms such as TRW are no better than the simple baseline estimation, in which $\hat{p}_i = 0.5$ for all nodes/variables i , for less than half of the full range of values of the sign probability q (i.e., for $q \in \{0.1, 0.4, 0.5, 0.6, 0.8, 0.9\}$). In fact, the performance of TRW is *almost exactly the same* as baseline across the range of non-extreme values of q . (Note how the plot of the values for trw and bl are essentially on top of each other for values of q other than 0 or 1.) On the other hand, note how fp (ce) is consistently better than bl across the whole range of values for q . In fact, fp (ce) is always in the set of (statistically) best performers for all q : i.e., the single best for $q = 1.0$, and indistinguishable from trw for $q = 0.0$; gs and bp for $q \in \{0.1, 0.5, 0.6, 0.7\}$; nr, gs, and bp for $q = 0.2$; gs for $q \in \{0.3, 0.4, 0.9\}$; and gs and mf for $q = 0.8$. The proposed fp (ce) is also best at both extremes, while trw is only best when all weights are negative. Almost all the methods other than fp (ce) are no better, and often worst, than bl, except for bp and trw for $q = 0.0$; trw for $q \in \{0.2, 0.7, 1.0\}$; trw and gs for $q = 0.3$; and gs for $q = 0.8$.

Fig. 3 plots the proportion of non-convergent runs of bp (higher curve) and trw (lower curve). Note the interesting behavior of bp: the likelihood of convergence diminishes considerably as q nears 0.5. The effect is almost symmetrical. In contrast, the effect on the non-convergence of trw is negligible. Note, however, that the bp’s non-convergence does not seem to really affect its performance in terms of marginal error. This plot provides additional evidence for our claim that the generative model of random Ising models used for evaluation does lead to harder problem instances.

²⁸Such class of models follows from our general experience with similar models. We find that instantiating Ising model parameters using densities over edge-weights tended to yield to relatively easier models than the ones we obtain by fixing the magnitude of the edge-weights and varying the probability of their sign, independently for each edge.

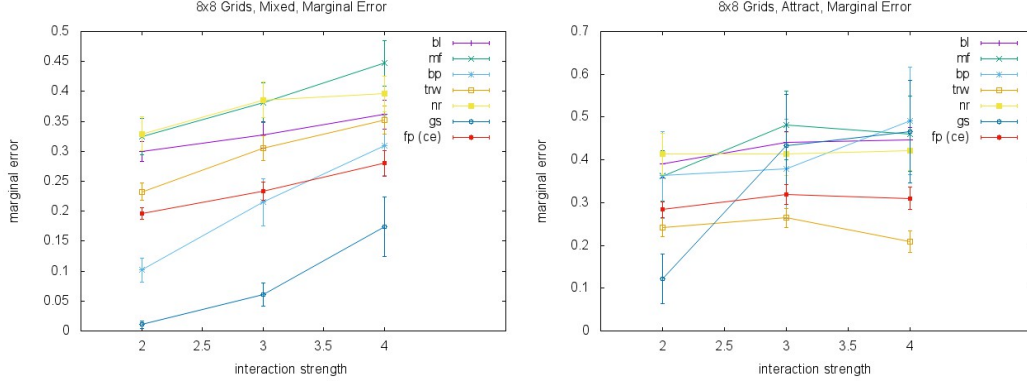


Figure 1: **Standard Evaluation on Ising Models with 8x8 Grids.** The left and right plots are for the so-called “mixed” and “attractive” instances of Ising models, respectively. For both plots, the x-axis is the largest magnitude of the edge-weights: i.e., $w = \max_{(i,j) \in E} |w_{ij}|$. The y-axis is the average, over 50 randomly generated Ising models, of the *average*, over all of the 64 variables, of the absolute difference between the estimate and exact marginal probability for the random variable corresponding to that node, along with their corresponding 95% confidence intervals (CIs). The legend in each plot is for different approximation algorithms: bl = baseline; mf = mean field; bp = belief propagation; trw = tree reweighed message-passing; nr = simple no-regret algorithm; gs = Gibbs sampler; and fp (ce) = the CE version of our version of the fictitious play for the 2-player potential game described in Section 3.4. We refer the reader to the main body for implementation details and a thorough discussion of the results.

5 Future Work and New Opportunities

It would be nice to have a better understanding of the exact relationship between the *true* joint distribution of the MRF and the equilibrium points of the induced graphical potential game. As an example of how this would be useful, it might give us a better idea as to whether one can think of a Gibbs sampler, or other Monte-Carlo sampling algorithms, as providing solutions to equilibrium problems of certain quality.

We still need to study the effect of the parameter d defining the grid graph for the Ising model on the effectiveness of all the approximation algorithms. We are currently setting up experiments on Ising models with a larger number of nodes. In particular, we are considering $d = 12$, which leads to $n = 144$ nodes/variables.

We also need to study the effect of the number of iterations m of the fp (ce) and nr on the quality of the estimates of the node-marginals, as well as the convergence rates. Also, we need to study the effect of m and d on algorithms such as the MSNE-instantiation of the fictitious-play heuristic we propose. Of course, we need to compare the results of such experimental studies with a similar study for gs; right now, we use 10^6 iterations for gs in our experiments. Similarly, we should study the effect of the stopping rules and maximum number of iterations for mf, bp, and trw on their marginal-error performance, as well as the effect of sequential vs. simultaneous updates for mf and bp on the same. We should also study more carefully the effect of varying or optimizing the edge-appearance parameters for trw on its marginal-error. Finally, we should empirically study the effect of “dampening” the updates for nr, bp, and trw on their performance.

The focus of the experimental evaluation in this paper was testing our proposed, game-theoretically-inspired algorithms for belief inference with standard algorithms in the literature of probabilistic

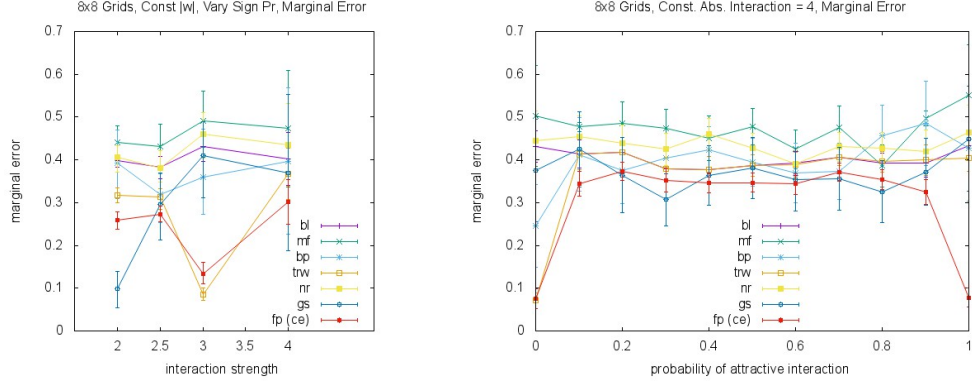


Figure 2: Evaluation on Ising Models with 8x8 Grids, Uniform Interaction Magnitude, and Varied Probability of Attractive Interactions. (*Left plot*) The x-axis, y-axis, and legend are as in Fig. 1, except the edge-weight *magnitude* w is constant for each interaction strength in the x-axis (i.e., 2, 2.5, 3, and 4), and nr uses 10^6 iterations. For all cases, the result is the average over all values of the probability of attractive interaction $q \in \{0.0, 0.1, \dots, 0.9, 1.0\}$, and over 5 Ising models for each q ; *except* for the case of constant edge-weight magnitude $w = 4$, in which case the average for each q is over 50 Ising-model samples. Said differently, the overall average for the cases of $w \in \{2.0, 2.5, 3.0\}$ is over a total of 55 Ising models, while those for the case of $w = 4$ is over a total of 550 models. Note that the standard 95% CIs based on a Gaussian approximation resulting from the Central Limit Theorem (CLT) do not directly apply here because the averages are over different q values, each of which may have different distributional properties (e.g., different variances). For $w < 4$, because we are computing the average marginal-error over every q , each based on only 5 samples, we use the bootstrap method to compute the 95% CIs over the overall average for each method and each w , using 100 samples. For $w = 4$, because we have 50 samples for each q , we use a properly adapted version of the standard 95% CIs which modifies the calculation of the overall variance to account for distributional differences from each q . (*Right plot*) Results for each q value with $w = 4$, with 50 models as samples for each, along with their corresponding individual 95% CIs computed as usual. We refer the reader to the main body for a thorough discussion.

graphical models with relatively “simple” implementations (e.g., do not require calls to software packages or the implementation of complex optimizations). An empirical study involving such algorithms with considerably more complex implementations must have a precise experimental methodology and design that accounts for not only the complexity of implementation, but also a fair comparison that achieves the right balance between measures of solution quality and running times. We leave such evaluations for future work because of the level of complexity required to carry them out correctly.

The work in this paper just “scratches the surface” in terms of the synergy between equilibrium computation in game theory and belief inference in probabilistic graphical models. We state and discuss several immediate theoretical, algorithmic, and computational implications, but many more may be possible. An even broader and more thorough literature review than the one provided in this manuscript is necessary to fully exploit this connection. Thus, many opportunities for novel contributions remain available in either direction.

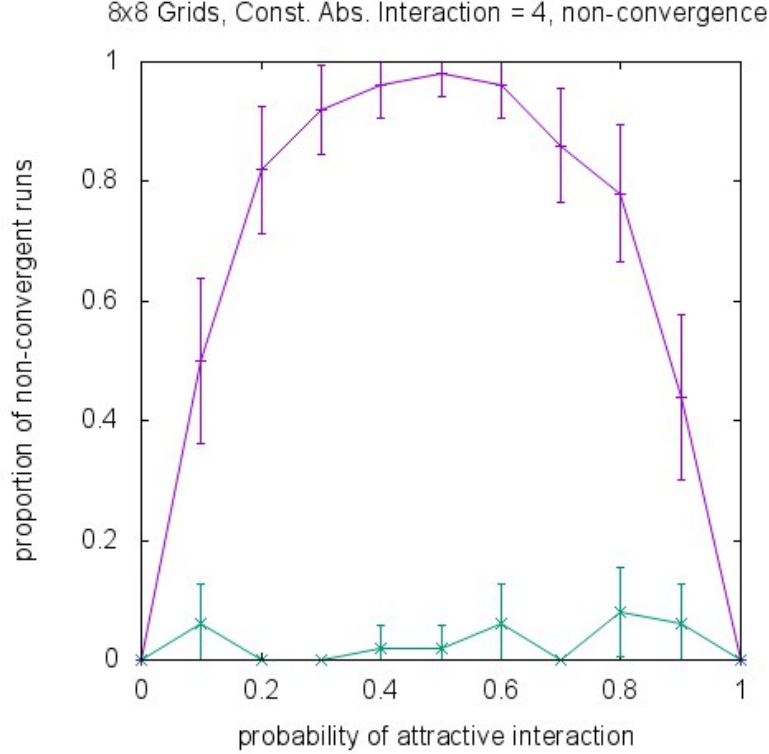


Figure 3: **Evaluation on Ising Models with 8x8 Grids, Uniform Interaction Magnitude ($w=4$), and Varied Probability of Attractive Interactions q : Proportion of Non-convergent BP and TRW Runs.** This plot shows proportion, along with standard individual 95% CIs, of non-convergent runs (y-axis) of bp (higher curve) and trw (lower curve), as a function of the probability of attractive interaction q (x-axis) for Ising models with constant edge-weights magnitude equal to 4. The setup is as described in the right plot of Fig. 2 for the case of edge-weight magnitude $w = 4$. The proportion is out of 50 runs for each $q \in \{0, 0.1, 0.2, \dots, 0.9, 1\}$. Note how the convergence of bp degrades when q nears 0.5. Note the almost symmetric effect on non-convergence for bp. Note also that bp non-convergence seems uncorrelated with its performance, as shown in Fig. 2 (Right plot). While trw may also show non-convergence outside non-uniform edge-weights, the effect is less drastic than for bp.

6 Contributions

We provide general formulations of the problem of inference in MRFs as equilibrium computation in graphical potential games. We provide connections, particularly to variational inference approaches, with immediate algorithmic, computational, and theoretical implications to belief inference in probabilistic graphical models that follow immediately from the game-theory literature to various related problems. We provide two approaches for approximate belief inference: a local and a global approach. We experimentally evaluate the effectiveness of the proposed algorithms in the context of Ising models with grid graphs, and provide a characterization of their computational effectiveness based on common measures used to characterize classes of Ising models (e.g., mixed and attractive models with different relative levels of magnitude between the edge weights and node bias values).

We also empirically evaluate effectiveness using a slightly different approach in which we keep the edge-weight magnitude constant but vary the “sign probability.” We show how most methods are often not much better than a simple baseline (i.e., estimate that the marginal probabilities are all equal to 0.5) in that class of Ising models. Our results suggest that the proposed class of Ising models does indeed lead to harder instances than the popular models used for empirical evaluation in the same context of Ising models. We empirically show that our proposed method based on a global approach is best, beating even TRW within that class, and shinning in a class of Ising models with constant, “highly attractive” edge-weights, in which it is often better than all other alternatives we evaluated. Note that TRW is generally considered state-of-the-art. We propose such class of Ising models for future evaluations because our experimental results suggest that instances from that class are often the hardest. While our more local approach is not as effective as our global approach or TRW, in fairness, almost all of the alternatives are often no better than a simple baseline: estimate the marginal probability to be 0.5.

In closing, our hope is that the work presented in this manuscript will start a conversation on the synergy between equilibrium computation and belief inference, which appears to be an intriguing and potentially fruitful research direction for both mathematical, algorithmic, and computational game-theory and probabilistic graphical models.

Acknowledgements

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²⁹See <http://www-personal.umd.umich.edu/~leortiz/papers/infeq.pdf> for the original manuscript, and http://www-personal.umd.umich.edu/~leortiz/infeq_info_v2.html for detailed information, including its history.

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